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The relation between the scaling of Husimi functions and the linear phase insensitive amplification of the corresponding quantum states and its implications

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Abstract. Using the description of the linear phase insensitive amplification of a quantum state available in the literature we prove that to every state $\rho(t)$, produced from the initial state $\rho(0)$ by such amplification, there corresponds a phase space distribution $\lambda^2 D_0(\lambda q, \lambda p)$, where $D_0(q, p)$ is the Husimi function of the initial state. The scaling parameter λ satisfies the relation $0 < \lambda < 1$ and decreases with time while, as we have shown earlier, the scaled function is again a Husimi distribution.

We prove, using these facts, that if the definition of the phase insensitive amplification is physically correct then the distribution of the phase of a state is unambiguously given as the corresponding marginal distribution obtained from its Husimi distribution represented in polar coordinates.

1. Introduction

The generally accepted definition of the phase distribution of a quantum state, although needed for various reasons, does not exist. The reason for this is that the problem of finding the proper dynamical variable corresponding to the phase of a quantum field is not solved. There have been numerous attempts to construct satisfactory phase operators and the whole question has been the subject of discussion and controversy for a long time [1–5]. Several attempts have been made to examine the adequacy of some definitions experimentally, but no clear conclusion was reached [6–8].

In another area of research, instead of a search for the phase operator, attempts have been made to define the distribution of the phase directly, with the help of functions appearing in phase space formulations of quantum mechanics [9, 10]. It is well known that, regarding the quantum mechanical state, all such functions are informationally complete [11]. The central question now is how to obtain the distribution of phase from such functions. The simplest way would be to define the distribution of phase, using the analogy with classical statistical mechanics, as the corresponding marginal distribution of the considered phase space function in polar coordinates in phase space. A new question then arises of which phase space function to use, because different phase space functions give different distributions of phase for the same quantum mechanical state, or whether to choose a completely different approach. The comparisons with experiment again did not give sufficient evidence to answer this question.

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In the present work we give the reasons for which one can assume that the phase distribution of quantum state is correctly given as the corresponding marginal distribution obtained from its Husimi function represented in polar coordinates in phase space.

Our arguments, which are given in the following sections, run as follows.

We first prove that in the class of Cohen distributions, which contains virtually all the most frequently used phase space functions, only the Husimi functions have the property to remain Husimi functions after the change of variables $(q, p) \to (\lambda q, \lambda p)$, $0 < \lambda < 1$.

Further, we relate the so-defined scaling of Husimi functions with the amplification of the state with the linear phase insensitive amplifier, as described by Glauber in [12–14]. We show that in the limit of the most quiet amplification the evolution of the initial state during the process of such amplification may be described as the scaling of the initial Husimi distribution. The scaling parameter λ is an exponentially decreasing function of the time duration of the process of amplification and can be made as small as we wish. We have proved in [15] that for sufficiently small λ the scaled Husimi functions begin to behave classically so that for such states all physical information may be obtained in a classical way. So, as the phase insensitive amplification does not change the phase of the initial state and the initial state in the process of amplification evolves to a classical state, we conclude that the distribution of phase of every state is given as a corresponding marginal of the Husimi function represented in polar coordinates. Finally, we discuss some consequences of our results.

2. Scaling invariance of the Husimi function and its consequences

In [15] we proved that after the transformation $(q, p) \to (\lambda q, \lambda p)$, where $0 < \lambda < 1$, every Husimi function with appropriate renormalization remains in the class of Husimi distributions. For completeness of further analysis we will first prove this fact here in a different, more constructive way.

Up to a numerical factor the Husimi distribution function may be defined as the diagonal matrix element of the density matrix $\hat{\rho}$ in the harmonic oscillator coherent state basis $|\alpha\rangle$ (hereafter $\hbar = 1$):

$$D_{\mathrm{H}}(q, p) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle. \tag{1}$$

Every density matrix may be represented in the form

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k| \qquad p_k \geqslant 0. \tag{2}$$

In the basis of the eigenenergy states of a harmonic oscillator, the coherent states $|\alpha\rangle$ are given as

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{3}$$

where

$$\alpha = \frac{1}{\sqrt{2}} \left(\sqrt{b} q + \frac{\mathrm{i}}{\sqrt{b}} p \right) \qquad b = m\omega.$$

From (1)–(3) we have

$$D_{\mathrm{H}}(q, p) = \mathrm{e}^{-|\alpha|^2} \sum_k p_k \left(\sum_n A_n^{k*} \frac{\alpha^{*n}}{\sqrt{n!}} \right) \left(\sum_m A_m^k \frac{\alpha^m}{\sqrt{m!}} \right) \tag{4}$$

where

$$A_n^k = \langle \psi_k | n \rangle.$$

After the change of variables $(q, p) \to (\lambda q, \lambda p)$ and the renormalization, we have

$$F(q, p) = \lambda^2 D_{\mathrm{H}}(\lambda q, \lambda p) = \frac{\lambda^2}{\pi} \mathrm{e}^{-\lambda^2|\alpha|^2} \sum_k p_k \left(\sum_n A_n^{k*} \frac{\lambda^n \alpha^{*n}}{\sqrt{n!}} \right) \left(\sum_m A_m^k \frac{\lambda^m \alpha^m}{\sqrt{m!}} \right). \tag{5}$$

Now

$$\mathrm{e}^{-\lambda^2|\alpha|^2} = \mathrm{e}^{-|\alpha|^2 + (1-\lambda^2)|\alpha|^2} = \mathrm{e}^{-|\alpha|^2} \sum_{s=0}^{\infty} \frac{(1-\lambda^2)^s}{s!} (\alpha\alpha^*)^s$$

so that we can write

$$F(q, p) = \lambda^2 \frac{\mathrm{e}^{-|\alpha|^2}}{\pi} \sum_{k,s} p_k \left[\left(\sum_n A_n^{k*} \lambda^n \frac{\alpha^{*(n+s)}}{\sqrt{n!}} \right) \left(\sum_m A_m^k \lambda^m \frac{\alpha^{m+s}}{\sqrt{m!}} \right) \frac{(1-\lambda^2)^s}{s!} \right]. \tag{6}$$

This expression may be written in the form

$$F(q, p) = \frac{1}{\pi} \langle \alpha | \tilde{\rho} | \alpha \rangle \tag{7}$$

where

$$\tilde{\rho} = \lambda^2 \sum_{k,s} p_k \left(\sum_n A_n^{k*} \lambda^n \frac{\sqrt{(n+s)!}}{\sqrt{n!}} |n+s\rangle \right) \times \left(\sum_m A_m^k \lambda^m \frac{\sqrt{(m+s)!}}{\sqrt{m!}} \langle m+s| \right) \times \frac{(1-\lambda^2)^s}{s!}. \tag{8}$$

For $0 < \lambda < 1$, $\tilde{\rho}$ is positive definite and since

$$\sum_{s=0}^{\infty} \frac{(1-\lambda^2)^2}{s!} \frac{(n+s)!}{n!} = \frac{1}{\lambda^{2n+2}} \tag{9}$$

it is normalized, so $\tilde{\rho}$ is a density matrix. Due to this, the function

$$F(q, p) = \lambda^2 D_{\mathrm{H}}(\lambda q, \lambda p) = \frac{1}{\lambda} \langle \alpha | \tilde{\rho} | \alpha \rangle \tag{10}$$

is again a Husimi distribution.

So, the class of Husimi distributions is scale invariant.

We will now prove that all the other most frequently used phase space functions are not scale invariant. To this end we will consider the class of Cohen functions [16]:

$$P_\psi(f; q, p) = \frac{1}{(2\pi)^2} \int \exp[-\mathrm{i}(\theta q + \tau p - \theta u) f(\theta, \tau) \psi^*(u - \tfrac{1}{2}\tau) \psi(a + \tfrac{1}{2}\tau) \, \mathrm{d}\theta \tau \, \mathrm{d}u. \tag{11}$$

Straightforward calculation shows that this expression may be written in the form

$$P_\psi(f; q, p) = \frac{1}{2\pi} \int \exp[-\mathrm{i}(\theta q + \tau p)] f(\theta, \tau) V(\theta, \tau) \, \mathrm{d}\theta \, \mathrm{d}\tau \tag{12}$$

where $V(\theta, \tau)$ is a Fourier transform of a Wigner function

$$V(\theta, \tau) = \frac{1}{2\pi} \int W(q, p) \mathrm{e}^{\mathrm{i}q\theta + \mathrm{i}p\tau} \, \mathrm{d}q \, \mathrm{d}p = \frac{1}{(2\pi)^2} \int \psi^*(q + \tfrac{1}{2}\xi) \psi(q - \tfrac{1}{2}\xi) \mathrm{e}^{\mathrm{i}p\xi} \mathrm{e}^{\mathrm{i}q\theta + \mathrm{i}p\tau} \, \mathrm{d}\xi \, \mathrm{d}q \, \mathrm{d}p. \tag{13}$$

We will restrict our considerations to the case when the function $f(\theta, \tau)$ may be represented in the form $f(\theta, \tau) = \mathrm{e}^{\Omega(\theta,\tau)}$, and Ω is an entire function

$$\Omega(\theta, \tau) = \sum_{n,m} A_{nm}\theta^n\tau^m. \tag{14}$$

Even with this restriction, the Cohen function contains the Husimi function, the Wigner function, the P function, the standard and the antistandard function and many others as special cases [17, 18].

Introducing the operator

$$\mathrm{e}^{\hat{\Omega}(q,p)} = \exp\left(\sum_{n,m} A_{nm}\mathrm{i}^{n+m}\frac{\partial^{n+m}}{\partial q^n\partial p^m}\right) \tag{15}$$

we can write

$$\exp[-\mathrm{i}(\theta q + \tau p)]f(\theta, \tau) = \mathrm{e}^{\hat{\Omega}}\mathrm{e}^{-\mathrm{i}(\theta q+\tau p)}. \tag{16}$$

Now the expression (12) may be written in the form

$$P_\psi(f; q, p) = \frac{1}{2\pi}\int \mathrm{e}^{\hat{\Omega}(\partial/\partial q,\partial/\partial p)}[\mathrm{e}^{-\mathrm{i}(\theta q+\tau p)}V(\theta, \tau)\,\mathrm{d}\theta\,\mathrm{d}\tau] \tag{17}$$

or, after the change of the order of integration and differentiation over the parameters, which are present in the exponent

$$P_\psi(f; q, p) = \mathrm{e}^{\hat{\Omega}(\partial/\partial q,\partial/\partial p)}W(q, p). \tag{18}$$

Between the Wigner function and the Husimi function exists the relation [17]

$$W(q, p) = \exp\left[-\frac{1}{4b}\frac{\partial^2}{\partial q^2} - \frac{b}{4}\frac{\partial^2}{\partial p^2}\right]D_{\mathrm{H}}(q, p). \tag{19}$$

From (18) and (19) follows the relation

$$P_\psi(f, q, p) = \mathrm{e}^{\hat{\Omega}(\partial/\partial q,\partial/\partial p)}\exp\left[-\frac{1}{4b}\frac{\partial^2}{\partial q^2} - \frac{b}{4}\frac{\partial^2}{\partial p^2}\right]D_{\mathrm{H}}(q, p). \tag{20}$$

Now making the change of variables $(q, p) \to (\lambda q, \lambda p)$, $0 < \lambda < 1$, we obtain

$$\begin{aligned} P_\psi(f; \lambda q, \lambda p) &= \exp\left[\sum_{nm}\frac{A_{nm}}{\lambda^{n+m}}\mathrm{i}^{n+m}\frac{\partial^{n+m}}{\partial q^n\partial p^m} - \frac{1}{4b\lambda^2}\frac{\partial^2}{\partial q^2} - \frac{b}{4\lambda^2}\frac{\partial^2}{\partial p^2}\right]D_{\mathrm{H}}(\lambda q, \lambda p) \\ &= \exp\left[{\sum_{nm}}'\frac{A_{nm}}{\lambda^{n+m}}\mathrm{i}^{n+m}\frac{\partial^{n+m}}{\partial q^n\partial p^m} + \frac{1}{\lambda^2}\left(A_{20} - \frac{1}{4b}\right)\frac{\partial^2}{\partial q^2}\right. \\ &\quad \left. + \frac{1}{\lambda^2}\left(A_{02} - \frac{b}{4}\right)\frac{\partial^2}{\partial p^2}\right]\cdot D_{\mathrm{H}}(\lambda q, \lambda p) \end{aligned} \tag{21}$$

where $\sum'_{n,m}$ denotes the sum from which the terms with A_{20} and A_{02} are excluded. One concrete class of the Cohen functions is characterized by the concrete choice of the parameters A_{mn}. We will now analyse the question of whether it can happen that for every λ and every fixed set of the parameters A_{mn}, one can always find another function from the same class which is equal to the scaled function under consideration. We have just shown that this is the case for the class of Husimi distributions. Now we shall prove that all the other classes of Cohen functions are not invariant under scaling. Namely, as $D_{\mathrm{H}}(\lambda q, \lambda p)$ in (21) is scale invariant, then if the function on the left-hand side in (21) were scale invariant for every λ and every state described by the corresponding Husimi distribution, it would be necessary that the following conditions be fulfilled:

$$\left(A_{20}-\frac{1}{4b}\right)\frac{1}{\lambda^2}=\left(A_{20}-\frac{1}{4b}\right) \qquad \left(A_{02}-\frac{b}{4}\right)\frac{1}{\lambda^2}=\left(A_{02}-\frac{b}{4}\right)$$
$$\frac{A_{mn}}{\lambda^{n+m}}=A_{mn}. \tag{22}$$

The last conditions follow from the comparison of (20) and (21). However, these conditions may be satisfied only if $\lambda = 1$, which is a trivial case, or when

$$A_{20}=\frac{1}{4b} \qquad A_{02}=\frac{b}{4} \tag{23}$$

and all other coefficients are zero and this is a class of Husimi distributions.

So, in every class of the Cohen functions, except the class of Husimi distributions, there always may be found at least one function such that after the scaling it does not remain in the same class of functions. In this respect the class of Husimi functions has a privileged position among the other most frequently used phase space functions.

Now we shall find the relation between the λ scaling of the Husimi function and the amplification of the state by a phase insensitive linear amplifier, as described by Glauber [12–14].

According to Glauber the amplification process forces the initial density matrix

$$\rho(0)=\int P(t=0;\alpha_0)|\alpha_0\rangle\langle\alpha_0|\mathrm{d}^2\alpha_0 \tag{24}$$

to evolve into

$$\rho(t)=\int P(t;\alpha)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha \tag{25}$$

with

$$P(t;\alpha)=\int \mathrm{d}^2\alpha_0 P(t;\alpha|t_0=0;\alpha_0)P(t_0=0;\alpha_0) \tag{26}$$

where $P(t_0 = 0; \alpha_0)$ is the Glauber P function of the initial state. The conditional probability under the integral is given by the following expression [13, 14]

$$P(t;\alpha|t_0=0;\alpha_0)=\pi^{-1}N(t)^{-1}\exp[-N^{-1}|\alpha-\alpha_0\mathrm{e}^{kt}|^2] \tag{27}$$

where

$$N(t)=(1+\langle n\rangle)(\mathrm{e}^{2kt}-1). \tag{28}$$

Here k and $\langle n\rangle$ denote the amplification constant and the mean number of photons of the heat bath, respectively.

We shall now describe the same process in terms of the Husimi distribution. By the definition of the Husimi function we have

$$D(t;q,p)=\frac{1}{\pi}\langle\alpha|\rho(t)|\alpha\rangle=\frac{1}{\pi}\int \mathrm{d}^2\alpha' P(t;\alpha')|\langle\alpha|\alpha'\rangle|^2. \tag{29}$$

After the integration over α' we obtain

$$D(t;q,p)=\frac{1}{2\pi}\frac{1}{M(t)}\int \mathrm{d}^2\alpha_0 P(t_0=0;\alpha_0)$$
$$\times\exp\left[-\frac{b}{2}\frac{1}{M(t)}(q-q_0\mathrm{e}^{kt})^2-\frac{1}{2b}\frac{1}{M(t)}(p-p_0\mathrm{e}^{kt})^2\right] \tag{30}$$

where $M(t) \equiv 1 + N(t)$.

In the limit of most quiet amplification when $\langle n \rangle = 0$ we have $1 + N = e^{2kt}$, and this expression simplifies to

$$D(t; q, p) = \frac{1}{2\pi} \int d^2\alpha_0 P(t_0 = 0; \alpha_0) \exp\left[-\frac{b}{2}(q e^{-kt} - q_0)^2 - \frac{1}{2b}(p e^{-kt} - p_0)^2\right] e^{-2kt}. \tag{31}$$

Since

$$D(0; q, p) \equiv D_0(q, p) = \frac{1}{2\pi} \int d^2\alpha_0 P(t_0 = 0; \alpha_0) \exp\left[-\frac{b}{2}(q - q_0)^2 - \frac{1}{2b}(p - p_0)^2\right] \tag{32}$$

we can write

$$D(t; q, p) = \lambda^2 D_0(\lambda q, \lambda p) \tag{33}$$

where $\lambda = e^{-kt}$, and obviously $0 < \lambda \leqslant 1$.

The last equality shows that the scaling transformation of Husimi function may be interpreted as the linear, most quiet amplification of the initial state, and *vice versa*. In this way we have established the important relation between the formal procedure of the scaling of Husimi distribution and the concrete physical process of the amplification of a quantum state.

Concluding this section let us note that for a state described by $\lambda^2 D_0(\lambda q, \lambda p)$ with sufficiently small λ, all the Cohen functions describing the same state, become close to each other, and in the limiting case $\lambda \to 0$, they become equal. This is so because all of them are related to $\lambda^2 D_0(\lambda q, \lambda p)$ through the relation of the type $\lambda^2(1 + \sum_{mn} A_{mn} \partial^{m+n}/\partial q^m \partial p^n) D_0(q, p)$ and every differentiation over q and p causes the multiplication by the small parameter λ.

3. Concluding remarks

When the scaling parameter λ is sufficiently small, the transformed Husimi distribution $\lambda^2 D(\lambda q, \lambda p)$ behaves almost as a classical distribution because it is a normalized non-negative function, and consequently a true probability distribution, and because, as we have shown in [15], the average values of all physical quantities in such a state may be obtained in the classical way so that the error introduced by such a procedure may be made negligible.

We will now discuss, in the light of our new results, the consequences of this fact regarding the problem of the definition of the phase distribution of a quantum state.

We proved in [15] that the function $\lambda^2 D(\lambda q, \lambda p)$, for every positive λ smaller than unity, represents some physical state. We have shown in the preceding section that all these states may be produced from the initial state $D(q, p)$ by the linear phase insensitive amplification and that this process may be described as a scaling transformation of the initial state.

Phase insensitive amplification, by definition, does not change the phase of the amplified state. By performing the amplification of the state for a sufficiently long time the scaling parameter λ may be made as small as we wish. This means that the amplified state begins to behave classically so that its phase distribution may be obtained in a classical way. Due to this, if the definition of the phase insensitive amplification is physically correct then the problem of the distribution of phase for any state of harmonic oscillator is solved in a unique way. The phase distribution of a state is then the distribution obtained after the integration of the corresponding Husimi function represented in polar coordinates over the polar radius.

When the considered state is quasiclassical its phase distribution may be obtained with a good approximation from any of the most frequently used phase space functions with the same procedure because, as we have shown, all such functions are nearly equal for such states. However, our results show that in the extreme quantum situations this procedure may be physically justified for Husimi functions only.

In the very vast literature related to the problem of phase of the harmonic oscillator, of which virtually all was described and discussed in the time honoured review by Carruthers and Nieto [3], and the recent review by Lynch [19], only in the paper of Schleich *et al* [20] was Glauber's approach of phase insensitive amplification related to the problem of the phase distribution. The main new result of this paper, according to its authors, is the conclusion that: 'In the limit of strong but quiet amplification, the P distribution of the final amplified state is identical to the Q function of the original unamplified state displaced by the number of photons fed in by the amplifier' (equation (10) of [20]). They also noted that 'the phase distribution obtained by integrating (over the radius) *the Q function of the initial state* is identical to that found by integrating the *P distribution of the final state*'.

One part of our new results may be considered as a generalization of the quoted results. We have succeeded in describing Glauber's process of the phase insensitive amplification completely in terms of a Husimi function and proved that the time evolution of a state in the process of the amplification may be represented as the monotonic continuous time-dependent scaling of the initial Husimi function. Such a relation of scaling was established in [20] only between the initial and the final strongly amplified almost classical state ($kt \gg 1$).

The second quoted result from [20] is also generalized by us in the following two respects. We have shown that for strongly amplified states all the phase space functions from the Cohen's class become asymptotically equal, so that by integrating in polar coordinates whichever represents the same state one will obtain practically the same results for the phase distribution as by integrating the P distribution, and all these results are asymptotically equal to the result obtained by integrating the Husimi function of the initial state. Our results also show that for Husimi distributions alone, the phase distributions obtained after integration in polar coordinates are exactly equal, not only for the initial and the strongly amplified final state, but also for all the states generated from the initial state in the process of the phase insensitive amplification at any moment of time. This unique property of Husimi distributions, unnoticed until now, is based on the fact established in section 2 that among all the Cohen's functions only the class of Husimi functions is closed under identical scaling of coordinate and momentum.

In this paper we have used the process of the linear phase insensitive amplification as part of the physical ground which has enabled us to reach and formulate explicitly the fundamental conclusion that the phase distribution of a quantum state is correctly defined as the corresponding marginal distribution in polar coordinates obtained from the Husimi function of the considered state.

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